Matching with Contracts

Hatfield and Milgrom (AER, Sep 2005) summary by N. Antić

Examines connection between two-sided matching and auction/contract theory. Echenique (2012, AER) shows this model can be embedded in the Kelso-Crawford (1982, ECTA) framework. The extension by Hatfiled-Kojima (2010, JET) does not embed in the older framework, thus this is a good workhorse model.

Basic Model

- Finite sets of doctors and hospitals D and H, respectively
- X is a finite *set of contracts* (generic)
  - A *contract* $x \in X$ matches a doctor, x_D, and hospital, x_H
  - x may contain more information, e.g., doctor's wage
  - $A \subset X$ is an *allocation* if $\forall (d,h), (d',h') \in A$, $d \neq d'$
- $\succ_d$ is a total order on $X_d \cup \{\emptyset\}$, $X_d := \{x \in X \mid x_D = d\}$
  - Contract x is *acceptable* to doctor d if $x \succ_d \emptyset$
  - Given $Y \subset X$, let $C_d(Y) = \max_{\succ_d}(X_d \cap Y \cup \{\emptyset\})$
    - $C_D(Y) = \cup_d C_d(Y)$ denotes accepted contracts
    - $R_D(Y) = Y \setminus C_D(Y)$ denotes rejected contracts
- $\forall Y \subset X$, hospital preferences are given by a choice correspondence, $C_h(Y) \subset Y_h := \{x \in Y \mid x_H = h\}$, where for any $x, x' \in C_h(Y)$, $x \neq x' \Rightarrow x_D \neq x'_D$
  - Let $C_H(Y) = \cup_h C_h(Y)$, $R_H(Y) = Y \setminus C_H(Y)$
- A set of contracts $A \subset X$ is a *stable allocation* if

S1. $C_D(A) = C_H(A) = A$, and

S2. $\nexists h \in H$, $Y \subset X$ s.t. $Y = C_h(A \cup Y) \subset C_D(A \cup Y)$

Theorem 1. *If $(X_D, X_H) \subset X^2$ solves the system:*

$$\begin{aligned} X_D &= X \setminus R_H(X_H), \\ X_H &= X \setminus R_D(X_D), \end{aligned}$$

then $A := X_H \cap X_D$ is a stable allocation and $A = C_H(X_H) = C_D(X_D)$. Conversely, for any stable A, there exist X_D, X_H satisfying the system such that $A = X_H \cap X_D$.

Proof. ($\Rightarrow$) $A := X_H \cap X_D = X_D \setminus R_D(X_D) = C_D(X_D)$; similarly $A = C_H(X_H)$. By revealed preference $A = C_D(X_D) = C_D(A) = C_H(A) = C_H(X_H)$, thus A satisfies S1. Fix h and some $Y_h \subset C_D(A \cup Y_h)$. Since $A = C_D(X_D)$, by doctors' revealed preferences, $Y_h \cap X_D \subset C_D(X_D)$ and thus

$$Y_h \cap R_D(X_D) = Y_h \cap X_D \cap R_D(X_D) \subset C_D(X_D) \cap R_D(X_D) = \emptyset.$$

Hence, $Y_h \subset X \setminus R_D(X_D) = X_H$ and so if $Y_h \neq C_h(A)$, then $Y_h \prec_h C_h(X_h) = C_h(A)$. Thus $Y_h \neq C_h(A \cup Y_h)$ and S2 holds.

($\Leftarrow$) Let A be a stable allocation. Note that $A = C_H(A) = C_D(A)$. Define:

$$\begin{aligned} X_H &= \bigcup_{d \in D} \{x \in X_d \cup \{\emptyset\} : x \succeq_d A_d\}, \\ X_D &= \bigcup_{d \in D} \{x \in X_d \cup \{\emptyset\} : x \preceq_d A_d\}. \end{aligned}$$

Clearly $A, X_D \setminus A, X_H \setminus A$ is a partition of X. If $C_h(X_H) \neq A_h$, then $Y = C_h(X_H)$ violates S2. Thus $C_H(X_H) = A$, since $C_h(X_H) = A_h$ for all h. Hence:

$$\begin{aligned} X \setminus R_H(X_H) &= X \setminus (X_H \setminus A) = X_D, \text{ and} \\ X \setminus R_D(X_D) &= X \setminus (X_D \setminus C_D(X_D)) = X_H, \end{aligned}$$

since by definition $C_D(X_D) = A$. □

- Elements of X are *substitutes* for h if $R_h(Y) \subset R_h(Z)$ for any $Y \subset Z \subset X$
  - i.e., if R_h is isotone (order-perserving) with resepct to $\subset$
  - New contracts make hospital weakly less interested in old
  - Coincides with demand theory definition in the appropriate setting (Theorem 2)

Generalized Deferred Acceptance (DA) Algorithm

- Generalized DA algorithm, $F : X \times X \to X \times X$

$$F(X_D, X_H) = (X \setminus R_H(X_H), X \setminus R_D(X \setminus R_H(X_H)))$$

  - R_D (by revealed preference) and R_H (by assumption) are isotone, and thus F is also isotone wrt $\geq$, defined as:

$$F(X_D, X_H) \geq F(Y_D, Y_H) \text{ iff } Y_D \subset X_D \text{ and } X_H \subset Y_H$$

Theorem (Tarski). *If $(L, \geq)$ is a complete lattice and $f : L \to L$ is isotone, then f has a fixed point. Further, if P is the set of fixed points of f, then $(P, \geq)$ is a complete lattice.*

- A special case of this is the existence theorem in the paper

Theorem 3. *Suppose contracts are substitutes for hospitals. Then:*

(a) The set of fixed points of F is a complete (finite) lattice

(b) Starting at $(X, \emptyset)$, iteration on F converges monotonically to the highest fixed point $(\overline{X}_D, \overline{X}_H)$

(c) Starting at $(\emptyset, X)$, iteration on F converges monotonically to the lowest fixed point $(\underline{X}_D, \underline{X}_H)$

- The highest [lowest] fixed point is the stable allocation most preferred by doctors [hospitals] (Theorem 4)
  - Starting at $(X, \emptyset)$ is equivalent to doctor-proposing DA
  - Starting at $(\emptyset, X)$ is equivalent to hospital-proposing DA
- To see equivalence with doctor-proposing DA:
  - Define set $X_H(t)$ as the cumulative contracts offered by doctors to hospitals up to iteration t
  - Define $X_D(t)$ as the cumulative set of contracts that have not yet been rejected by hospitals up to iteration t
  - $X_H(t) \cap X_D(t)$ are contracts conditionally accepted
  - Using the definition of F, we have that:

$$\begin{aligned} X_D(t) = X \setminus R_H(X_H(t-1)), \\ X_H(t) = X \setminus R_D(X_D(t)). \end{aligned} \tag{1}$$

- If $|H| \geq 2$, and R_h is not isotone for some h, then there exists a preference profile for doctors and a preference profile for another hospital h', which has a single job opening, such that no stable match exists (Theorem 5)

- i.e., substitutes assumption almost needed for existence

Properties of Generalized DA

- Vacancy chain dynamics can be described by adjusting F to reject all offers made to a retiring doctor
  - After a doctor retires, a new stable allocation is achieved by iteration on the adjusted F; unretired doctors weakly better off and hospitals weakly worse off (Theorem 6)
- The preferences of hospital $h \in H$ satisfy the *law of aggregate demand* if for all $Y \subset Z$, $|C_h(Y)| \leq |C_h(Z)|$
  - If more contracts are offered, each hospital should demand weakly more contracts
  - This can be characterized nicely in the matching with wages setting:

Theorem 7. *If hospital h's preferences are quasi-linear and satisfy the substitutes condition, then they satisfy the law of aggregate demand.*

- A rural hospital theorem also holds in matching with contracts

Theorem 8. *If hospital preferences satisfy substitutes and the law of aggregate demand, then for every stable allocation* (X_D, X_H) *and every* $d \in D$ *and* $h \in H$, $|C_d(X_D)| = \left|C_d(\overline{X}_D)\right|$ *and* $|C_h(X_H)| = \left|C_h(\overline{X}_H)\right|$.

- Hospitals and doctors always get the same number of contracts
  - Necessity of law of aggregate demand for this result:

Theorem 9. *If there exists a hospital h, sets* $Y \subset Z \subset X$ *such that* $|C_h(Y)| > |C_h(Z)|$, *and* $|H| > 1$, *then there exist singleton preferences for the other hospitals and doctors such that the number of doctors employed by h is different for two stable matches.*

- Truth-telling is optimal in a few senses
- Let hospitals' preferences satisfy substitutes and the law of aggregate demand, and let the matching algorithm produce the doctor-optimal match. Then, fixing the preferences of the other doctors and of all the hospitals, let x be the contract that d obtains by submitting the set of preferences Pd : z1 d
- Let hospital h have preferences such that $|C_h(X)| > |C_h(X \cup \{x\})|$ and let there exist two contracts y, z such that $y_D \neq x_D \neq z_D$ and y, $z \in R_h(X \cup \{x\}) - R_h(X)$. Then if another hospital h' exists, there exist singleton preferences for the hospitals besides h and preferences for the doctors such that it is not a dominant strategy for all doctors to reveal their preferences truthfully.

Relationship to Proxy Auctions

- The algorithm described by 1 may not converge if substitutes assumption fails, even if a fixed point exists
- The following algorithm is yet another characterization of DA and is suitable for the non-substitutes case, when there is just one hospital (the auctoneer)

$$\begin{aligned} X_D(t) &= X \smallsetminus R_H(X_H(t-1)), \\ X_H(t) &= X_H(t-1) \cup C_D(X_D(t)). \end{aligned} \quad (2)$$

Theorem 15. *Under the substitute assumption, with* $X_D(0) = X$ *and* $X_H(0) = \varnothing$, *the sequences of pairs* $\{(X_D(t), X_H(t))\}$ *generated by the two laws of motion 1 and 2 are identical.*

- The difference between the 1 and 2 is the second equation
  - In 2, $X_H(t)$ monotonic even if R_h is not isotone

Theorem 16. *When the doctor-offering cumulative offer process with a single hospital terminates at time t with outcome* $(X_D(t), X_H(t))$, *the hospital's choice* $C_H(X_H(t))$ *is a stable collection of contracts.*

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